

# A variant of Hörmander's $L^2$ theorem for Dirac operator in Clifford analysis

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**Abstract** In this paper, we give the Hörmander's  $L^2$  theorem for Dirac operator over an open subset  $\Omega \in \mathbb{R}^{n+1}$  with Clifford algebra. Some sufficient condition on the existence of the weak solutions for Dirac operator has been found in the sense of Clifford analysis. In particular, if  $\Omega$  is bounded, then we prove that for any  $f$  in  $L^2$  space with value in Clifford algebra, there exists a weak solution of Dirac operator such that

$$\overline{D}u = f$$

with  $u$  in the  $L^2$  space as well. The method is based on Hörmander's  $L^2$  existence theorem in complex analysis and the  $L^2$  weighted space is utilised.

**Keywords** Hörmander's  $L^2$  theorem · Clifford analysis · weak solution · Dirac operator

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## 1 Introduction

The development of function theories on Clifford algebras has proved a useful setting for generalizing many aspects of one variable complex function theory to higher dimensions. The study of these function theories is referred to as Clifford analysis [Brackx et al(1982),Huang et al(2006),Gong et al(2009),Ryan(2000)], which is closely related to a number of studies made in mathematical physics, and many applications in this area have been found in recent years. In [Ryan(1995)], Ryan considered solutions of the polynomial Dirac operator, which afforded an integral representation. Furthermore, the author gave a Pompeiu representation for  $C^1$ -functions in a Lipschitz bounded domain. In [Ryan(1990)], the author presented a classification of linear, conformally invariant, Clifford-algebra-valued differential operators over  $\mathbb{C}^n$ , which comprised the Dirac operator and its iterates. In [Qian and Ryan(1996)], Qian and Ryan used Vahlen matrices to study the conformal covariance of various types of Hardy spaces over hypersurfaces in  $\mathbb{R}^n$ . In [De Ridder et al(2012)], the discrete Fueter polynomials was introduced, which formed a basis of the space of discrete spherical monogenics. Moreover, the explicit construction for this discrete Fueter basis, in arbitrary dimension  $m$  and for arbitrary homogeneity degree  $k$  was presented as well.

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In [Hörmander(1965)], the famous Hörmander's  $L^2$  existence and approximation theorems was given for the  $\bar{\partial}$  operator in pseudo-convex domains in  $\mathbb{C}^n$ . When  $n = 1$ , the existence theorem of complex variable can be deduced. The aim of this paper is to establish a Hörmander's  $L^2$  theorem in  $\mathbb{R}^{n+1}$  with Clifford analysis, and present sufficient condition on the existence of the weak solutions for Dirac operator in the sense of Clifford algebra.

Let  $\mathcal{A}$  be a real Clifford algebra over an  $(n+1)$ -dimensional real vector space  $\mathbb{R}^{n+1}$  and the corresponding norm on  $\mathcal{A}$  is given by  $|\lambda|_0 = \sqrt{(\lambda, \lambda)_0}$  (see subsection 2.1). Let  $\Omega$  be an open subset of  $\mathbb{R}^{n+1}$ ,  $L^2(\Omega, \mathcal{A}, \varphi)$  be a right Hilbert  $\mathcal{A}$ -module for a given function  $\varphi \in C^2(\Omega, \mathbb{R})$  with the norm given by Definition 29. (see subsection 2.3).  $\bar{D}$  denotes the Dirac differential operator and the dual operator  $\bar{D}_\varphi^*$  of  $\bar{D}$  is given by (4). For  $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ ,  $\Delta = \sum_{i=0}^n \frac{\partial^2}{\partial x_i^2}$ . Then we can obtain our main results as follows.

**Theorem 11** *Given  $f \in L^2(\Omega, \mathcal{A}, \varphi)$ , there exists  $u \in L^2(\Omega, \mathcal{A}, \varphi)$  such that*

$$\bar{D}u = f \quad (1)$$

*with*

$$\|u\|^2 = \int_{\Omega} |u|_0^2 e^{-\varphi} dx \leq 2^{2n} c \quad (2)$$

*if*

$$|(f, \alpha)_\varphi|_0^2 \leq c \|\bar{D}_\varphi^* \alpha\|^2 = c \int_{\Omega} |\bar{D}_\varphi^* \alpha|_0^2 e^{-\varphi} dx, \quad \forall \alpha \in C_0^\infty(\Omega, \mathcal{A}). \quad (3)$$

*Conversely, if there exists  $u \in L^2(\Omega, \mathcal{A}, \varphi)$  such that (1) is satisfied with*

$$\|u\|^2 \leq c$$

*Then we can get the inequality (3) for norm estimation.*

The factor  $2^{2n}$  in (2) comes from the definition of the norm in Clifford analysis. If  $n = 1$ , then the factor would disappear which gives a necessary and sufficient condition in the theorem. From the above theorem, we give the following sufficient condition on the existence of weak solutions for Dirac operator.

**Theorem 12** *Given  $\varphi \in C^2(\Omega, \mathbb{R})$  and  $n > 1$ ;  $\Delta\varphi \geq 0$ , and  $\frac{\partial^2 \varphi}{\partial x_j \partial x_i} = 0$ ,  $i \neq j$ ,  $1 \leq i, j \leq n$  and  $\frac{\partial^2 \varphi}{\partial x_i^2} \leq 0$ ,  $1 \leq i \leq n$ . Then for all  $f \in L^2(\Omega, \mathcal{A}, \varphi)$  with  $\int_{\Omega} \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx = c < \infty$ , there exists a  $u \in L^2(\Omega, \mathcal{A}, \varphi)$  such that*

$$\bar{D}u = f$$

*with*

$$\|u\|^2 = \int_{\Omega} |u|_0^2 e^{-\varphi} dx \leq 2^{2n} \int_{\Omega} \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx.$$

**Remark 13** *Assuming  $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ , it is easy to see that  $\varphi(x) = x_0^2$  satisfies the conditions in Theorem 12. Another simple example would be*

$$\varphi(x) = (n+1)x_0^2 - \sum_{i=1}^n x_i^2.$$

*It is obvious that  $\Delta\varphi(x) = 2$ ,  $\frac{\partial^2 \varphi}{\partial x_i^2} = -2$ , and  $\frac{\partial^2 \varphi}{\partial x_j \partial x_i} = 0$ ,  $i \neq j$ ,  $1 \leq i, j \leq n$ .*

**Corollary 14** *Given  $\varphi \in C^2(\Omega, \mathbb{R})$ , and  $\varphi(x) = \varphi(x_0)$  with  $\varphi''(x_0) \geq 0$ . Then for all  $f \in L^2(\Omega, \mathcal{A}, \varphi)$  with  $\int_{\Omega} \frac{|f|_0^2}{\varphi''} e^{-\varphi} dx = c < \infty$ , there exists a  $u \in L^2(\Omega, \mathcal{A}, \varphi)$  such that*

$$\overline{D}u = f$$

with

$$\|u\|^2 = \int_{\Omega} |u|_0^2 e^{-\varphi} dx \leq 2^{2n} \int_{\Omega} \frac{|f|_0^2}{\varphi''} e^{-\varphi} dx.$$

It is noticed that there is nothing to do with the boundary conditions of  $\Omega$  in the above results. This phenomenon is totally different with the famous Hörmander's  $L^2$  existence theorems of several complex variables in [Hörmander(1965)]. Then we can also have the following theorem on global solutions.

**Theorem 15** *Given  $\varphi \in C^2(\mathbb{R}^{n+1}, \mathbb{R})$  with all derivative conditions in Theorem 11 satisfied. Then for all  $f \in L^2(\mathbb{R}^{n+1}, \mathcal{A}, \varphi)$  with  $\int_{\mathbb{R}^{n+1}} \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx = c < \infty$ , there exists a  $u \in L^2(\mathbb{R}^{n+1}, \mathcal{A}, \varphi)$  satisfying*

$$\overline{D}u = f$$

with

$$\|u\|^2 = \int_{\mathbb{R}^{n+1}} |u|_0^2 e^{-\varphi} dx \leq 2^{2n} \int_{\mathbb{R}^{n+1}} \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx.$$

On the other hand, if the boundary of  $\Omega$  is concerned, we consider a special kind of domain  $\Omega_0 = \{x \in \mathbb{R}^{n+1} : a \leq x_0 \leq b\}$  for any  $a, b \in \mathbb{R}$  with  $a < b$ , then we can get the following theorem within  $L^2$  space instead of  $L^2$  weighted space.

**Theorem 16** *Let  $f \in L^2(\Omega_0, \mathcal{A})$ . Then there exists a  $u \in L^2(\Omega_0, \mathcal{A})$  such that*

$$\overline{D}u = f$$

with

$$\int_{\Omega_0} |u|_0^2 dx \leq 2^{2n} c(a, b) \int_{\Omega_0} |f|_0^2 dx$$

and  $c(a, b)$  is a factor depending on  $a, b$ .

*Proof* Let  $\varphi(x) = x_0^2$ . It can be obtained that  $L^2(\Omega_0, \mathcal{A}) = L^2(\Omega_0, \mathcal{A}, \varphi)$  for the boundary of  $x_0$ . Then the theorem is proved with Theorem 12.

**Remark 17** *In particular, any bounded domain  $\Omega$  in  $\mathbb{R}^{n+1}$  can be regarded as one type of  $\Omega_0$ . Therefore, it comes from Theorem 16 that for any  $f \in L^2(\Omega, \mathcal{A})$ , we can find a weak solution of Dirac operator  $\overline{D}u = f$  with  $u \in L^2(\Omega, \mathcal{A})$ .*

## 2 Preliminaries

To make the paper self-contained, some basic notations and results used in this paper are included.

### 2.1 The Clifford algebra $\mathcal{A}$

Let  $\mathcal{A}$  be a real Clifford algebra over an  $(n+1)$ -dimensional real vector space  $\mathbb{R}^{n+1}$  with orthogonal basis  $e := \{e_0, e_1, \dots, e_n\}$ , where  $e_0 = 1$  is a unit element in  $\mathbb{R}^{n+1}$ . Furthermore,

$$\begin{cases} e_i e_j + e_j e_i = 0, & i \neq j \\ e_i^2 = -1, & i = 1, \dots, n. \end{cases}$$

Then  $\mathcal{A}$  has its basis

$$\{e_A = e_{h_1} \dots e_{h_r} = e_{h_1} \dots e_{h_r} : 1 \leq h_1 < \dots < h_r \leq n, 1 \leq r \leq n\}.$$

If  $i \in \{h_1, \dots, h_r\}$ , we denote  $i \in A$  and if  $i \notin \{h_1, \dots, h_r\}$ , we denote  $i \notin A$ .  $A - i$  means  $\{h_1, \dots, h_r\} \setminus \{i\}$  and  $A + i$  means  $\{h_1, \dots, h_r\} \cup \{i\}$ . So the real Clifford algebra is composed of elements having the type  $a = \sum_A x_A e_A$ , in which  $x_A \in \mathbb{R}$  are real numbers.

For  $a \in \mathcal{A}$ , we give the inversion in the Clifford algebra as follows:  $a^* = \sum_A x_A e_A^*$

where  $e_A^* = (-1)^{|A|} e_A$  and  $|A| = n(A)$  is the  $r \in \mathbb{Z}^+$  as  $e_A = e_{h_1 \dots h_r}$ . When  $A = \emptyset$ ,  $e_A = e_0$ ,  $|A| = 0$ . Next, we define the reversion in the Clifford algebra, which is given by  $a^\dagger = \sum_A x_A e_A^\dagger$  where  $e_A^\dagger = (-1)^{(|A|-1)|A|/2} e_A$ . Now we present the involution which is a combination of the inversion and the reversion introduced above.

$$\bar{a} = \sum_A x_A \bar{e}_A$$

where  $\bar{e}_A = e_A^{*\dagger} = (-1)^{(|A|+1)|A|/2} e_A$ . From the definition, one can easily deduce that  $e_A \bar{e}_A = \bar{e}_A e_A = 1$ . Furthermore, we have

$$\overline{\lambda\mu} = \bar{\mu}\bar{\lambda}, \quad \forall \lambda, \mu \in \mathcal{A}.$$

Let  $a = \sum_A x_A e_A$  be a Clifford number. The coefficient  $x_A$  of the  $e_A$ -component will also be denoted by  $[a]_A$ . In particular the coefficient  $x_0$  of the  $e_0$ -component will be denoted by  $[a]_0$ , which is called the scalar part of the Clifford number  $a$ . An inner product on  $\mathcal{A}$  is defined by putting for any  $\lambda, \mu \in \mathcal{A}$ ,  $(\lambda, \mu)_0 := 2^n [\lambda \bar{\mu}]_0 = 2^n \sum_A \lambda_A \mu_A$ .

The corresponding norm on  $\mathcal{A}$  reads  $|\lambda|_0 = \sqrt{(\lambda, \lambda)_0}$ .

We define a real functional on  $\mathcal{A}$  that  $\tau_{e_A} : \mathcal{A} \rightarrow \mathbb{R}$

$$\langle \tau_{e_A}, \mu \rangle = 2^n (-1)^{(|A|+1)|A|/2} \mu_A.$$

In the special case where  $A = \emptyset$  we have

$$\langle \tau_{e_0}, \mu \rangle = 2^n \mu_0.$$

Let  $\Omega$  be an open subset of  $\mathbb{R}^{n+1}$ . Then functions  $f$  defined in  $\Omega$  and with values in  $\mathcal{A}$  are considered. They are of the form

$$f(x) = \sum_A f_A(x) e_A$$

where  $f_A(x)$  are functions with real value. Let  $\bar{D}$  denotes the Dirac differential operator

$$\bar{D} = \sum_{i=0}^n e_i \partial_{x_i},$$

its action on functions from the left and from the right being governed by the rules

$$\bar{D}f = \sum_{i,A} e_i e_A \partial_{x_i} f_A \quad \text{and} \quad f\bar{D} = \sum_{i,A} e_A e_i \partial_{x_i} f_A.$$

$f$  is called left-monogenic if  $\bar{D}f = 0$  and it is called right-monogenic if  $f\bar{D} = 0$ . The conjugate operator is given by

$$D = \sum_{i=0}^n \bar{e}_i \partial_{x_i}.$$

It can be found that

$$\bar{D}D = D\bar{D} = \Delta$$

where  $\Delta$  denotes the classical Laplacian in  $\mathbb{R}^{n+1}$ . When  $n = 1$ , one can think of  $x_0$  as the real part and of  $x_1$  as the imaginary part of the variable and to identify  $e_1$  with  $i$ . the operator  $\bar{D}$  then take the form  $\bar{D} = \partial_{x_0} + i\partial_{x_1}$ , which is similar with the operator  $\bar{\partial}$  in complex analysis.

## 2.2 Modules over Clifford algebras

This subsection is to give some general information concerning a class of topological modules over Clifford algebras. In the sequel definitions and properties will be stated for left  $\mathcal{A}$ -module and their duals, the passage to the case of right  $\mathcal{A}$ -module being straight-forward.

**Definition 21 (unitary left  $\mathcal{A}$ -module)** Let  $X$  be a unitary left  $\mathcal{A}$ -module, i.e.  $X$  is abelian group and a law  $(\lambda, f) \rightarrow \lambda f : \mathcal{A} \times X \rightarrow X$  is defined such that  $\forall \lambda, \mu \in \mathcal{A}$ , and  $f, g \in X$

- (1)  $(\lambda + \mu)f = \lambda f + \mu f$ ,
- (2)  $\lambda \mu f = \lambda(\mu f)$ ,
- (3)  $\lambda(f + g) = \lambda f + \lambda g$ ,
- (4)  $e_0 f = f$ .

Moreover, when speaking of a submodule  $E$  of the unitary left  $\mathcal{A}$ -module  $X$ , we mean that  $E$  is a non empty subset of  $X$  which becomes a unitary left  $\mathcal{A}$ -module too when restricting the module operations of  $X$  to  $E$ .

**Definition 22 (left  $\mathcal{A}$ -linear operator)** If  $X, Y$  are unitary left  $\mathcal{A}$ -modules, then  $T : X \rightarrow Y$  is said to be a left  $\mathcal{A}$ -linear operator, if  $\forall f, g \in X$  and  $\lambda \in \mathcal{A}$

$$T(\lambda f + g) = \lambda T(f) + T(g).$$

The set of all “ $T$ ” is denoted by  $L(X, Y)$ . If  $Y = \mathcal{A}$ ,  $L(X, \mathcal{A})$  is called the algebraic dual of  $X$  and denoted by  $X^{*alg}$ . Its elements are called left  $\mathcal{A}$ -linear functionals on  $X$  and for any  $T \in X^{*alg}$  and  $f \in X$ , we denote by  $\langle T, f \rangle$  the value of  $T$  at  $f$ .

**Definition 23 (bounded functional)** An element  $T \in X^{*alg}$  is called bounded, if there exist a semi-norm  $p$  on  $X$  and  $c > 0$  such that for all  $f \in X$

$$|\langle T, f \rangle|_0 \leq c \cdot p(f).$$

**Theorem 24 (Hahn-Banach type theorem)**[Brackx et al(1982)] Let  $X$  be a unitary left  $\mathcal{A}$ -module with semi-norm  $p$ ,  $Y$  be a submodule of  $X$ , and  $T$  be a left  $\mathcal{A}$ -linear functional on  $Y$  such that for some  $c > 0$ ,

$$|\langle T, g \rangle|_0 \leq c \cdot p(g), \quad \forall g \in Y$$

Then there exists a left  $\mathcal{A}$ -linear functional  $\tilde{T}$  on  $X$  such that

- (1)  $\tilde{T}|_Y = T$ ,
- (2) for some  $c^* > 0$ ,  $|\langle \tilde{T}, f \rangle|_0 \leq c^* \cdot p(f)$ ,  $\forall f \in X$ .

**Definition 25 (inner product on a unitary right  $\mathcal{A}$ -module)** Let  $H$  be a unitary right  $\mathcal{A}$ -module, then a function  $(\cdot, \cdot) : H \times H \rightarrow \mathcal{A}$  is said to be a inner product on  $H$  if for all  $f, g, h \in H$  and  $\lambda \in \mathcal{A}$ ,

- (1)  $(f, g + h) = (f, g) + (f, h)$ ,
- (2)  $(f, g\lambda) = (f, g)\lambda$ ,
- (3)  $(f, g) = \overline{(g, f)}$ ,
- (4)  $\langle \tau_{e_0}, (f, f) \rangle \geq 0$  and  $\langle \tau_{e_0}, (f, f) \rangle = 0$  if and only if  $f = 0$ ,
- (5)  $\langle \tau_{e_0}, (f\lambda, f\lambda) \rangle \leq |\lambda|_0^2 \langle \tau_{e_0}, (f, f) \rangle$ .

From the definition on inner product, putting for each  $f \in H$

$$\|f\|^2 = \langle \tau_{e_0}, (f, f) \rangle,$$

then it can be obtained that for any  $f, g \in H$ ,

$$|\langle \tau_{e_0}, (f, g) \rangle| \leq \|f\| \|g\|, \quad \|f + g\| \leq \|f\| + \|g\|.$$

Hence,  $\|\cdot\|$  is a proper norm on  $H$  turning it into a normed right  $\mathcal{A}$ -module. Moreover, we have the following Cauchy-Schwarz inequality.

**Proposition 26** [Brackx et al(1982)] For all  $f, g \in H$ ,  $|(f, g)|_0 \leq \|f\| \|g\|$ .

**Definition 27 (right Hilbert  $\mathcal{A}$ -module)** Let  $H$  be a unitary right  $\mathcal{A}$ -module provided with an inner product  $(\cdot, \cdot)$ . Then it is called a right Hilbert  $\mathcal{A}$ -module if it is complete for the norm topology derived from the inner product.

**Theorem 28 (Riesz representation theorem)** [Brackx et al(1982)] Let  $H$  be a right Hilbert  $\mathcal{A}$ -modules and  $T \in H^{*alg}$ . Then  $T$  is bounded if and only if there exists a (unique) element  $g \in H$  such that for all  $f \in H$ ,

$$T(f) := \langle T, f \rangle = (g, f).$$

### 2.3 Hilbert space of square integrable functions

Now we extend the standard Hilbert space of square integrable functions to Clifford algebra. First, we denote  $L^1(\Omega, \mu)$  and  $L^2(\Omega, \mu)$  be the sets of all integrable or square integrable functions defined on the domain  $\Omega \in \mathbb{R}^{n+1}$  with respect to the measure  $\mu$ . Then  $L^1(\Omega, \mathcal{A}, \mu)$  and  $L^2(\Omega, \mathcal{A}, \mu)$  are defined as the sets of functions  $f : \Omega \rightarrow \mathcal{A}$  which are integrable or square integrable with respect to  $\mu$ , i.e., if  $f = \sum_A f_A e_A$ , then for each  $A$ ,  $f_A \in L^1(\Omega, \mu)$  and  $f_A^2 \in L^1(\Omega, \mu)$ , respectively. Then **one may easily check that  $L^1(\Omega, \mathcal{A}, \mu)$  and  $L^2(\Omega, \mathcal{A}, \mu)$  are unitary bi- $\mathcal{A}$ -module, i.e., unitary left- $\mathcal{A}$ -module and unitary right- $\mathcal{A}$ -module.** Furthermore, for any  $f, g \in L^2(\Omega, \mathcal{A}, \mu)$ ,  $\bar{f} \in L^2(\Omega, \mathcal{A}, \mu)$  while  $\bar{f}g \in L^1(\Omega, \mathcal{A}, \mu)$ , where  $\bar{f}(x) = \overline{f(x)}$  and  $(\bar{f}g)(x) = \bar{f}(x)g(x)$ ,  $x \in \Omega$ . Consider as a right  $\mathcal{A}$ -module, define for  $f, g \in L^2(\Omega, \mathcal{A}, \mu)$  that

$$(f, g) = \int_{\Omega} \bar{f}(x)g(x)d\mu.$$

Furthermore for any real linear functional  $T$  on  $\mathcal{A}$

$$\langle T, (f, g) \rangle = \langle T, \int_{\Omega} \bar{f}(x)g(x)d\mu \rangle = \int_{\Omega} \langle T, \bar{f}(x)g(x) \rangle d\mu.$$

Consequently, taking  $T = \tau_{e_0}$  we find that

$$\begin{aligned} \langle \tau_{e_0}, (f, f) \rangle &= \langle \tau_{e_0}, \int_{\Omega} \bar{f}(x)f(x)d\mu \rangle = \int_{\Omega} \langle \tau_{e_0}, \bar{f}(x)f(x) \rangle d\mu \\ &= \int_{\Omega} |f(x)|_0^2 d\mu. \end{aligned}$$

Hence, for all  $f \in L^2(\Omega, \mathcal{A}, \mu)$ ,  $\langle \tau_{e_0}, (f, f) \rangle \geq 0$  and  $\langle \tau_{e_0}, (f, f) \rangle = 0$  if and only if  $f = 0$  a.e. in  $\Omega$ . Then it is easy to see that under the inner product defined, all conditions for  $L^2(\Omega, \mathcal{A}, \mu)$  to be a unitary right inner product  $\mathcal{A}$ -module are satisfied. Since  $L^2(\Omega, \mathcal{A}, \mu) = \prod_A L^2(\Omega, \mu)$ , we have that  $L^2(\Omega, \mathcal{A}, \mu)$  is complete; in other words  $L^2(\Omega, \mathcal{A}, \mu)$  is a right Hilbert  $\mathcal{A}$ -module, with the norm

$$\|f\|^2 = \langle \tau_{e_0}, (f, f) \rangle = \int_{\Omega} |f(x)|_0^2 d\mu$$

for  $f \in L^2(\Omega, \mathcal{A}, \mu)$ .

**Definition 29 (weighted  $L^2$  space)** Similar with  $L^2(\Omega, \mathcal{A}, \mu)$ , we can define the weighted  $L^2(H, \mathcal{A}, \varphi)$  for a given function  $\varphi \in C^2(\Omega, \mathbb{R})$ . First, let

$$L^2(\Omega, \varphi) = \{f|f : \Omega \rightarrow \mathbb{R}, \int_{\Omega} |f(x)|^2 e^{-\varphi} dx < +\infty\}.$$

Then we denote

$$L^2(H, \mathcal{A}, \varphi) = \{f|f : \Omega \rightarrow \mathcal{A}, f = \sum_A f_A e_A, f_A \in L^2(\Omega, \varphi)\}.$$

Moreover, for all  $f, g \in L^2(H, \mathcal{A}, \varphi)$ , we define

$$(f, g)_\varphi = \int_{\Omega} \bar{f}(x)g(x)e^{-\varphi} dx.$$

Then it is also easy to see  $L^2(\Omega, \mathcal{A}, \varphi)$  is a right Hilbert  $\mathcal{A}$ -module, with the norm

$$\|f\|^2 = \langle \tau_{e_0}, (f, f)_\varphi \rangle = \int_{\Omega} |f(x)|_0^2 e^{-\varphi} dx$$

for  $f \in L^2(\Omega, \mathcal{A}, \varphi)$ .

## 2.4 Cauchy's integral formula

Let  $M$  be an  $(n+1)$ -dimensional differentiable and oriented manifold contained in some open subset  $\Sigma$  of  $\mathbb{R}^{n+1}$ . By means of the  $n$ -forms

$$d\hat{x}_i = dx_0 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n, \quad i = 0, 1, \dots, n,$$

an  $\mathcal{A}$ -valued  $n$ -form is introduced by putting

$$d\sigma = \sum_{i=0}^n (-1)^i e_i d\hat{x}_i,$$

similarly, denote

$$d\bar{\sigma} = \sum_{i=0}^n (-1)^i \bar{e}_i d\hat{x}_i.$$

Furthermore the volume-element

$$dx = dx_0 \wedge \cdots \wedge dx_n$$

is used.

**Proposition 210 (Stokes-Green Theorem)** [Brackx et al(1982)] If  $f, g \in C^1(\Sigma, \mathcal{A})$  then for any  $(n+1)$ -chain  $\Omega$  on  $M \subset \Sigma$ ,

$$\begin{aligned} \int_{\partial\Omega} f d\sigma g &= \int_{\Omega} (f \bar{D})g dx + \int_{\Omega} f (\bar{D}g) dx, \\ \int_{\partial\Omega} f d\bar{\sigma} g &= \int_{\Omega} (f D)g dx + \int_{\Omega} f (Dg) dx. \end{aligned}$$

**Remark 211** Denote  $C_0^\infty(\Omega, \mathbb{R})$  as the set of all smooth real-valued functions with compact support in  $\Omega$  and  $C_0^\infty(\Omega, \mathcal{A}) := \{f|f : \Omega \rightarrow \mathcal{A}, f = \sum_A f_A e_A, f_A \in C_0^\infty(\Omega, \mathbb{R})\}$ .

If  $f$  or  $g \in C_0^\infty(\Omega, \mathcal{A})$ , then we have from the Stokes-Green theorem that

$$\begin{aligned} \int_{\Omega} (f \bar{D})g dx &= - \int_{\Omega} f (\bar{D}g) dx, \\ \int_{\Omega} (f D)g dx &= - \int_{\Omega} f (Dg) dx. \end{aligned}$$

**Lemma 212** If  $u(x) \in C^1(\Omega, \mathcal{A})$ , then  $\overline{\bar{D}u} = \bar{u}D$ .

*Proof* Let  $u(x) = \sum_A e_A u_A$ . Then

$$\overline{\bar{D}u} = \sum_{i,A} \bar{e}_i \bar{e}_A \partial_{x_i} u_A = \sum_{i,A} \bar{e}_A \bar{e}_i \partial_{x_i} u_A = \bar{u}D.$$

**Lemma 213** [Huang et al(2006)] If  $u(x) = \sum_A e_A u_A$ ,  $v(x) = \sum_{i=1}^n e_i v_i$ , then

$$\bar{D}(uv) = (\bar{D}u)v + u(\bar{D}v) + \sum_{j=1}^n (e_j u - u e_j) \partial_{x_j} v.$$

## 2.5 Weak solutions

Let  $L_{loc}^1(\Omega, \mathcal{A}) := \{f|f : \Omega \rightarrow \mathcal{A}, f = \sum_A f_A e_A, f_A \in L_{loc}^1(\Omega, \mathbb{R})\}$ . Then we define the weak solution in the sense of Clifford algebra as follows.

**Definition 214** ( $\overline{D}$  solution in weak sense) *If  $f \in L_{loc}^1(\Omega, \mathcal{A})$ ,  $u : \Omega \rightarrow \mathcal{A}$  is a weak solution of*

$$\overline{D}u = f \text{ (or } Du = f)$$

*if for any  $\alpha \in C_0^\infty(\Omega, \mathcal{A})$ ,*

$$\int_{\Omega} \alpha f dx = - \int_{\Omega} (\alpha \overline{D})u dx \text{ (or } \int_{\Omega} \alpha f dx = - \int_{\Omega} (\alpha D)u dx).$$

It should be noticed that if  $u$  is a weak solution of Dirac equation  $\overline{D}u = 0$ , in addition, if  $u$  is smooth in  $\Omega$ , then it is left-monogenic. Now it is natural to give the definition of  $\Delta$  solution in the weak sense.

**Definition 215** ( $\Delta$  solution in weak sense) *If  $f \in L_{loc}^1(\Omega, \mathcal{A})$ ,  $u : \Omega \rightarrow \mathcal{A}$  is a weak solution of*

$$\Delta u = f$$

*if for any  $\alpha \in C_0^\infty(\Omega, \mathcal{A})$ ,*

$$\int_{\Omega} \alpha f dx = \int_{\Omega} (\Delta \alpha)u dx.$$

**Theorem 216** *If  $f \in L_{loc}^1(\Omega, \mathcal{A})$ , and  $\overline{D}f = 0$  in weak sense, then  $f$  is left-monogenic at any point of  $\Omega$ .*

*Proof :* Since  $\overline{D}f = 0$  in weak sense, then  $\Delta f = 0$  in weak sense. By Weyl's lemma,  $f$  is smooth in  $\Omega$  and has  $\Delta f = 0$  in classical sense, then of course  $f$  is left-monogenic at any point of  $\Omega$ .

**Remark 217** *This is useful to deal with uniqueness of weak solutions. for example, if  $u, v \in L_{loc}^1(\Omega, \mathcal{A})$  are two weak solutions of  $\overline{D}u = f$ , then  $u = v + w$  with any  $w$  left-monogenic.*

**Remark 218** *An important example of a left monogenic function is the generalized Cauchy kernel*

$$G(x) = \frac{1}{\omega_{n+1}} \frac{\overline{x}}{|x|^{n+1}},$$

where  $\omega_{n+1}$  denotes the surface area of the unit ball in  $\mathbb{R}^{n+1}$ . This function obviously belongs to  $L_{loc}^1(\Omega, \mathcal{A})$  and is a fundamental solution of the Dirac equation in the classical sense at any point of  $\mathbb{R}^{n+1}$  except 0. However, it is not a weak solution of the Dirac operator. In fact, if it satisfies  $\overline{D}f = 0$  in the weak sense, then from Theorem 216, it must be left-monogenic in the any point of  $\Omega$  which could include 0. Therefore, we get a contradiction.

For  $f \in L^2(\Omega, \mathcal{A}, \varphi)$ ,  $u : \Omega \rightarrow \mathcal{A}$ . If  $\overline{D}u = f$ , based on the Stokes-Green theorem, we can define the dual operator  $\overline{D}_\varphi^*$  of  $\overline{D}$  under the inner product of  $L^2(\Omega, \mathcal{A}, \varphi)$ . For any  $\alpha \in C_0^\infty(\Omega, \mathcal{A})$ ,

$$\begin{aligned} (\alpha, f)_\varphi &= \int_{\Omega} \bar{\alpha} f e^{-\varphi} dx = \int_{\Omega} \bar{\alpha} e^{-\varphi} f dx \\ &= \int_{\Omega} (\bar{\alpha} e^{-\varphi}) (\overline{D}u) dx \\ &= - \int_{\Omega} ((\bar{\alpha} e^{-\varphi}) \overline{D}) u dx \\ &= - \int_{\Omega} ((\bar{\alpha} e^{-\varphi}) \overline{D}) e^\varphi u e^{-\varphi} dx \\ &= \int_{\Omega} \overline{-e^\varphi D(\alpha e^{-\varphi})} u e^{-\varphi} dx \\ &= (-e^{-\varphi} D(\alpha e^{-\varphi}), u)_\varphi \triangleq (\overline{D}_\varphi^* \alpha, u)_\varphi, \end{aligned} \tag{4}$$



where  $\overline{D}_\varphi^* \alpha = -e^\varphi D(\alpha e^{-\varphi}) = \alpha(D\varphi) - D\alpha$ , i.e.

$$(\alpha, \overline{D}u)_\varphi = (\overline{D}_\varphi^* \alpha, u)_\varphi.$$

In the same way, we also have

$$(\overline{D}u, \alpha)_\varphi = (u, \overline{D}_\varphi^* \alpha)_\varphi.$$

### 3 The proof of Theorem 11

Now we are in the position of proving Theorem 11.

*Proof (Sufficiency)* From the definition of dual operator and Cauchy-Schwarz inequality in Proposition 26, we have

$$\begin{aligned} |(f, \alpha)_\varphi|_0^2 &= |(\overline{D}u, \alpha)_\varphi|_0^2 = |(u, \overline{D}_\varphi^* \alpha)_\varphi|_0^2 \\ &\leq \|u\|^2 \cdot \|\overline{D}_\varphi^* \alpha\|^2 \\ &\leq c \cdot \|\overline{D}_\varphi^* \alpha\|^2. \end{aligned}$$

(*necessity*) We aim to prove the necessity with Riesz representation theorem. First, we denote the submodule

$$E = \{\overline{D}_\varphi^* \alpha, \alpha \in C_0^\infty(\Omega, \mathcal{A}), \varphi \in C^2(\Omega, \mathbb{R})\} \subset L^2(\Omega, \mathcal{A}, \varphi).$$

Then we define a linear functional  $L_f$  on  $E$ , i.e.,  $L_f \in E^{*alg}$  for a fixed  $f \in L^2(\Omega, \mathcal{A}, \varphi)$  as follows,

$$\langle L_f, \overline{D}_\varphi^* \alpha \rangle = (f, \alpha)_\varphi = \int_\Omega \bar{f} \cdot \alpha \cdot e^{-\varphi} dx \in \mathcal{A}.$$

From (3), we have

$$|\langle L_f, \overline{D}_\varphi^* \alpha \rangle|_0 = |(f, \alpha)_\varphi|_0 \leq \sqrt{c} \cdot \|\overline{D}_\varphi^* \alpha\|,$$

which means that  $L_f$  is a bounded functional from Definition 23. By the Hahn-Banach type theorem in Theorem 24,  $L_f$  can be extended to a linear functional  $\tilde{L}_f$  on  $L^2(\Omega, \mathcal{A}, \varphi)$ , and with

$$|\langle \tilde{L}_f, g \rangle|_0 \leq \sqrt{c^*} \|g\|, \quad \forall g \in L^2(\Omega, \mathcal{A}, \varphi), \quad (5)$$

where  $\sqrt{c^*} = \sqrt{c} \cdot |e_0|_0$ , since  $|e_A|_0 = 2^{n/2}$ , then  $c^* = 2^n c$  from [Brackx et al(1982)]. Now we are in the position to use the Riesz representation theorem for the operator  $\tilde{L}_f$ . From Theorem 28, there exists a  $u \in L^2(\Omega, \mathcal{A}, \varphi)$  such that

$$\langle \tilde{L}_f, g \rangle = (u, g)_\varphi, \quad \forall g \in L^2(\Omega, \mathcal{A}, \varphi). \quad (6)$$

For  $\forall \alpha \in C_0^\infty(\Omega, \mathcal{A})$ , let  $g = \overline{D}_\varphi^* \alpha$ . Then

$$(f, \alpha)_\varphi = \langle \tilde{L}_f, \overline{D}_\varphi^* \alpha \rangle = (u, \overline{D}_\varphi^* \alpha)_\varphi = (\overline{D}u, \alpha)_\varphi,$$

which deduces that

$$\int_\Omega \bar{f} \alpha e^{-\varphi} dx = \int_\Omega \overline{(\overline{D}u)} \alpha e^{-\varphi} dx.$$

Conjugating both sides of above equation leads to

$$\int_\Omega \bar{\alpha} f \cdot e^{-\varphi} dx = \int_\Omega \bar{\alpha} (\overline{D}u) e^{-\varphi} dx.$$

Let  $\alpha = \bar{\alpha} e^\varphi$ , it can be obtained that

$$\int_\Omega \alpha f dx = \int_\Omega \alpha (\overline{D}u) dx, \quad \forall \alpha \in C_0^\infty(\Omega, \mathcal{A}).$$

Therefore,

$$\overline{D}u = f$$

is proved from the definition of weak solutions.

Next, we give the bound for the norm of  $u$ . Let  $g = u = \sum_A e_A u_A \in L^2(\Omega, \mathcal{A}, \varphi)$ , from (5) and (6), we get that

$$|(u, u)_\varphi|_0 \leq \sqrt{c^*} \|u\|. \quad (7)$$

On the other hand,

$$\begin{aligned} |(u, u)_\varphi|_0^2 &= \left| \int_\Omega \bar{u} u e^{-\varphi} dx \right|_0^2 \\ &= 2^n \cdot \left[ \int_\Omega \bar{u} u e^{-\varphi} dx \cdot \overline{\int_\Omega \bar{u} u e^{-\varphi} dx} \right]_0 \\ &= 2^n \left[ \int_\Omega \left( \sum_A u_A^2 + \sum_{A \neq B} \bar{e}_A e_B u_A u_B \right) e^{-\varphi} dx \cdot \overline{\int_\Omega \left( \sum_A u_A^2 + \sum_{A \neq B} \bar{e}_A e_B u_A u_B \right) e^{-\varphi} dx} \right]_0 \\ &= 2^n \left[ \left( \int_\Omega \sum_A u_A^2 e^{-\varphi} dx \right)^2 + \left( \int_\Omega \sum_{A \neq B} u_A u_B e^{-\varphi} dx \right)^2 \right], \end{aligned}$$

and

$$\|u\|^2 = \int_\Omega |u|_0^2 e^{-\varphi} dx = 2^n \int_\Omega [\bar{u} u]_0 e^{-\varphi} dx = 2^n \int_\Omega \sum_A u_A^2 \cdot e^{-\varphi} dx$$

So we have  $\|u\|^4 = 2^{2n} \cdot \left( \int_\Omega \sum_A u_A^2 \cdot e^{-\varphi} dx \right)^2$ . Hence,

$$|(u, u)_\varphi|_0^2 = 2^n \left[ \left( \int_\Omega \sum_A u_A^2 \cdot e^{-\varphi} dx \right)^2 + \left( \int_\Omega \sum_{A \neq B} u_A u_B e^{-\varphi} dx \right)^2 \right] \geq 2^{-n} \|u\|^4.$$

Combining with (7), it is obtained that

$$\|u\|^2 \leq 2^{n/2} |(u, u)_\varphi|_0 \leq 2^{n/2} \sqrt{c^*} \|u\|,$$

and

$$\|u\|^2 \leq 2^{2n} c.$$

The proof is completed.

#### 4 The proof of Theorem 12

It should be noticed that inequality (3) in Theorem 11 is related with  $\alpha \in C_0^\infty(\Omega, \mathcal{A})$ . In the following, we will give another sufficient condition that has nothing to do with the space  $C_0^\infty(\Omega, \mathcal{A})$ . First, we need to compute the norm of  $\|\overline{D}_\varphi^* \alpha\|$  for any  $\alpha \in C_0^\infty(\Omega, \mathcal{A})$ .

$$\begin{aligned}
\|\overline{D}_\varphi^* \alpha\|^2 &= \int_{\Omega} |\overline{D}_\varphi^* \alpha|_0^2 e^{-\varphi} dx \\
&= \int_{\Omega} \langle \tau_{e_0}, \overline{\overline{D}_\varphi^* \alpha} \cdot \overline{D}_\varphi^* \alpha \rangle e^{-\varphi} dx \\
&= \langle \tau_{e_0}, \int_{\Omega} \overline{\overline{D}_\varphi^* \alpha} \cdot \overline{D}_\varphi^* \alpha e^{-\varphi} dx \rangle \\
&= \langle \tau_{e_0}, (\overline{D}_\varphi^* \alpha, \overline{D}_\varphi^* \alpha)_\varphi \rangle \\
&= \langle \tau_{e_0}, (\alpha, \overline{D} \overline{D}_\varphi^* \alpha)_\varphi \rangle \\
&= \langle \tau_{e_0}, (\alpha, \overline{D}(\alpha(D\varphi) - D\alpha))_\varphi \rangle \\
&= \langle \tau_{e_0}, (\alpha, \overline{D}\alpha(D\varphi) + \alpha\Delta\varphi - \Delta\alpha + \sum_{j=1}^n (e_j\alpha - \alpha e_j) \frac{\partial}{\partial x_j}(D\varphi))_\varphi \rangle \\
&= \langle \tau_{e_0}, (\alpha, \overline{D}_\varphi^*(\overline{D}\alpha) + \alpha\Delta\varphi + \sum_{j=1}^n (e_j\alpha - \alpha e_j) \frac{\partial}{\partial x_j}(D\varphi))_\varphi \rangle \\
&= \langle \tau_{e_0}, (\alpha, \overline{D}_\varphi^*(\overline{D}\alpha))_\varphi + (\alpha, \alpha\Delta\varphi)_\varphi + (\alpha, \sum_{j=1}^n (e_j\alpha - \alpha e_j) \frac{\partial}{\partial x_j}(D\varphi))_\varphi \rangle \\
&= \langle \tau_{e_0}, (\alpha, \overline{D}_\varphi^*(\overline{D}\alpha))_\varphi \rangle + \langle \tau_{e_0}, (\alpha, \alpha\Delta\varphi)_\varphi \rangle + \langle \tau_{e_0}, (\alpha, \sum_{j=1}^n (e_j\alpha - \alpha e_j) \frac{\partial}{\partial x_j}(D\varphi))_\varphi \rangle \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \langle \tau_{e_0}, (\alpha, \overline{D}_\varphi^*(\overline{D}\alpha))_\varphi \rangle = \langle \tau_{e_0}, (\overline{D}\alpha, \overline{D}\alpha)_\varphi \rangle = \|\overline{D}\alpha\|^2, \\
I_2 &= \langle \tau_{e_0}, (\alpha, \alpha\Delta\varphi)_\varphi \rangle = \int_{\Omega} |\alpha|_0^2 \Delta\varphi e^{-\varphi} dx,
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \langle \tau_{e_0}, (\alpha, \sum_{j=1}^n (e_j\alpha - \alpha e_j) \frac{\partial}{\partial x_j}(D\varphi))_\varphi \rangle \\
&= \langle \tau_{e_0}, (\alpha, \sum_{j=1}^n (e_j\alpha - \alpha e_j) \frac{\partial}{\partial x_j} (\sum_{i=0}^n \bar{e}_i \frac{\partial \varphi}{\partial x_i}))_\varphi \rangle \\
&= \langle \tau_{e_0}, (\alpha, \sum_{j=1}^n \sum_{i=0}^n (e_j\alpha \bar{e}_i - \alpha e_j \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_j \partial x_i})_\varphi \rangle \\
&= \langle \tau_{e_0}, \int_{\Omega} \bar{\alpha} \sum_{j=1}^n \sum_{i=0}^n (e_j\alpha \bar{e}_i - \alpha e_j \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_j \partial x_i} e^{-\varphi} dx \rangle \\
&= \int_{\Omega} \langle \tau_{e_0}, \bar{\alpha} \sum_{j=1}^n \sum_{i=0}^n (e_j\alpha \bar{e}_i - \alpha e_j \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \rangle e^{-\varphi} dx.
\end{aligned}$$

It should be noticed that if  $n = 1$ , i.e., the space  $\mathbb{R}^2$  is considered, then  $I_3 = 0$ .

Since for  $1 \leq i, j \leq n$  and  $i \neq j$ ,  $e_j \bar{e}_i = -e_j e_i = e_i e_j = -e_i \bar{e}_j$ . For simplicity, let

$$\begin{aligned}
I_4 &= \langle \tau_{e_0}, \bar{\alpha} \sum_{j=1}^n \sum_{i=0}^n (e_j \alpha \bar{e}_i - \alpha e_j \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \rangle \\
&= \langle \tau_{e_0}, \sum_{j=1}^n \sum_{i=1}^n (\bar{\alpha} e_j \alpha \bar{e}_i - \bar{\alpha} \alpha e_j \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \rangle + \langle \tau_{e_0}, \sum_{j=1}^n (\bar{\alpha} e_j \alpha \bar{e}_0 - \bar{\alpha} \alpha e_j \bar{e}_0) \frac{\partial^2 \varphi}{\partial x_j \partial x_0} \rangle \\
&= \langle \tau_{e_0}, \sum_{i=1}^n (\bar{\alpha} e_i \alpha \bar{e}_i - \bar{\alpha} \alpha e_i \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle + \langle \tau_{e_0}, \sum_{j \neq i}^n (\bar{\alpha} e_j \alpha \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \rangle \\
&\quad + \langle \tau_{e_0}, \sum_{j=1}^n (\bar{\alpha} e_j \alpha \bar{e}_0 - \bar{\alpha} \alpha e_j \bar{e}_0) \frac{\partial^2 \varphi}{\partial x_j \partial x_0} \rangle \\
&= \langle \tau_{e_0}, \sum_{i=1}^n (\bar{\alpha} e_i \alpha \bar{e}_i - \bar{\alpha} \alpha) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle + \langle \tau_{e_0}, \sum_{j \neq i}^n (\bar{\alpha} e_j \alpha \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \rangle \\
&\quad + \langle \tau_{e_0}, \sum_{j=1}^n (\bar{\alpha} e_j \alpha \bar{e}_0 - \bar{\alpha} \alpha e_j \bar{e}_0) \frac{\partial^2 \varphi}{\partial x_j \partial x_0} \rangle \\
&= I_5 + I_6 + I_7.
\end{aligned}$$

Assume  $\alpha = \sum_A \alpha_A e_A \in \mathcal{A}$ ,  $\bar{\alpha} = \sum_A \alpha_A \bar{e}_A$ , then for any  $1 \leq i \leq n$ ,

$$\begin{aligned}
\bar{\alpha} e_i \alpha \bar{e}_i &= \sum_A \alpha_A \bar{e}_A e_i \cdot \sum_A \alpha_A e_A \bar{e}_i \\
&= \sum_A (-1)^{\frac{|A|(|A|+1)}{2}} \alpha_A e_A e_i \cdot \sum_A (-1) \alpha_A e_A \bar{e}_i
\end{aligned}$$

Therefore

$$\begin{aligned}
I_5 &= \langle \tau_{e_0}, \sum_{i=1}^n (\bar{\alpha} e_i \alpha \bar{e}_i - \bar{\alpha} \alpha) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle \\
&= \langle \tau_{e_0}, \sum_{i=1}^n (\bar{\alpha} e_i \alpha \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle - \langle \tau_{e_0}, \sum_{i=1}^n (\bar{\alpha} \alpha) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle \\
&= \langle \tau_{e_0}, \sum_{i=1}^n \left( \sum_A (-1)^{\frac{|A|(|A|+1)}{2}} \alpha_A e_A e_i \cdot \sum_A (-1) \alpha_A e_A e_i \right) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle - \langle \tau_{e_0}, \sum_{i=1}^n (\bar{\alpha} \alpha) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle \\
&= 2^n \sum_{i=1}^n \left( \sum_A (-1)^{\frac{|A|(|A|+1)}{2}+1} \alpha_A^2 e_A e_i e_A e_i \right) \frac{\partial^2 \varphi}{\partial x_i^2} - \sum_{i=1}^n |\alpha|_0^2 \frac{\partial^2 \varphi}{\partial x_i^2} \\
&= 2^n \sum_{i=1}^n \left( \sum_{i \notin A} (-1)^{\frac{|A|(|A|+1)}{2}+1} \alpha_A^2 \cdot \overline{e_A e_i} \cdot e_A e_i \cdot (-1)^{\frac{(|A|+1)(|A|+2)}{2}} \right. \\
&\quad \left. + \sum_{i \in A} (-1)^{\frac{|A|(|A|+1)}{2}+1} \cdot \alpha_A^2 \cdot \overline{e_{A-i}} \cdot e_{A-i} \cdot (-1)^{\frac{(|A|-1)(|A|)}{2}} \right) \frac{\partial^2 \varphi}{\partial x_i^2} - \sum_{i=1}^n |\alpha|_0^2 \frac{\partial^2 \varphi}{\partial x_i^2} \quad (8) \\
&= 2^n \sum_{i=1}^n \left( \sum_{i \notin A} (-1)^{\frac{|A|(|A|+1)}{2}+1+\frac{(|A|+1)(|A|+2)}{2}} \cdot \alpha_A^2 \right. \\
&\quad \left. + \sum_{i \in A} (-1)^{\frac{|A|(|A|+1)}{2}+1+\frac{(|A|-1)(|A|)}{2}} \cdot \alpha_A^2 \right) \frac{\partial^2 \varphi}{\partial x_i^2} - \sum_{i=1}^n |\alpha|_0^2 \frac{\partial^2 \varphi}{\partial x_i^2} \\
&= 2^n \sum_{i=1}^n \left( \sum_{i \notin A} (-1)^{|A|^2} \cdot \alpha_A^2 + \sum_{i \in A} (-1)^{|A|^2+1} \cdot \alpha_A^2 \right) \frac{\partial^2 \varphi}{\partial x_i^2} - \sum_{i=1}^n |\alpha|_0^2 \frac{\partial^2 \varphi}{\partial x_i^2} \\
&= 2^n \sum_{i=1}^n \left( \sum_{i \notin A, |A|^2 \text{ is odd}} (-2) \alpha_A^2 + \sum_{i \in A, |A|^2 \text{ is even}} (-2) \alpha_A^2 \right) \frac{\partial^2 \varphi}{\partial x_i^2} \\
&= -2^{n+1} \sum_{i=1}^n \left( \sum_{i \notin A, |A|^2 \text{ is odd}} \alpha_A^2 + \sum_{i \in A, |A|^2 \text{ is even}} \alpha_A^2 \right) \frac{\partial^2 \varphi}{\partial x_i^2}.
\end{aligned}$$

To consider  $I_7$ , we first study  $\bar{\alpha} e_j \alpha$  for any  $1 \leq j \leq n$ . Without loss of generality, let  $e_j = e_1$ ,  $\bar{\alpha} = \sum_A \alpha_A \bar{e}_A$ ,  $\alpha = \sum_A \alpha_A e_A$ . Then  $\bar{\alpha} e_1 \alpha = (\sum_A \alpha_A \bar{e}_A) e_1 (\sum_A \alpha_A e_A)$ .

When  $e_A = e_1 e_{h_2} e_{h_3} \cdots e_{h_r}$ , where  $1 < h_2 < h_3 < \cdots < h_r$  and  $1 < r \leq n$ .

$$\begin{aligned}
\alpha_A \bar{e}_A e_1 &= \alpha_{1h_2 \cdots h_r} (-1)^{\frac{r(r+1)}{2}} \cdot e_1 e_{h_2} e_{h_3} \cdots e_{h_r} \cdot e_1 \\
&= \alpha_{1h_2 \cdots h_r} (-1)^{\frac{r(r+1)}{2}+r} e_{h_2} e_{h_3} \cdots e_{h_r} \\
\alpha_A e_A e_1 &= \alpha_{1h_2 \cdots h_r} e_1 e_{h_2} \cdots e_{h_r} \cdot e_1 = \alpha_{1h_2 \cdots h_r} (-1)^r e_{h_2} \cdots e_{h_r}.
\end{aligned} \quad (9)$$

When  $e_A = e_1$ ,

$$\begin{aligned}
\alpha_A \bar{e}_A e_1 &= \alpha_1 \\
\alpha_A e_A e_1 &= -\alpha_1.
\end{aligned} \quad (10)$$

When  $e_A = e_{h_2} e_{h_3} \cdots e_{h_r}$ , where  $1 < h_2 < h_3 < \cdots < h_r$  and  $1 < r \leq n$ .

$$\begin{aligned}
\alpha_A \bar{e}_A e_1 &= \alpha_{h_2 \cdots h_r} (-1)^{\frac{(r-1)(r)}{2}} \cdot e_{h_2} e_{h_3} \cdots e_{h_r} \cdot e_1 \\
&= \alpha_{h_2 \cdots h_r} (-1)^{\frac{(r-1)(r)}{2}+r-1} e_1 e_{h_2} \cdots e_{h_r} \\
\alpha_A e_A e_1 &= \alpha_{h_2 \cdots h_r} e_{h_2} \cdots e_{h_r} \cdot e_1 = \alpha_{h_2 \cdots h_r} (-1)^{r-1} e_1 e_{h_2} \cdots e_{h_r}.
\end{aligned} \quad (11)$$

When  $e_A = e_0$ ,

$$\begin{aligned}
\alpha_A \bar{e}_A e_1 &= \alpha_0 e_1 \\
\alpha_A e_A e_1 &= \alpha_0 e_1.
\end{aligned} \quad (12)$$

To compute  $I_7$ , one needs to know the coefficient for  $e_0$  of  $\bar{\alpha}e_1\alpha - \bar{\alpha}\alpha e_1$ . It means that we should find out the corresponding terms of  $e_1e_{h_2}e_{h_3}\cdots e_{h_r}$  and  $e_{h_2}\cdots e_{h_r}$  in  $\bar{\alpha}e_1$  and  $\alpha$ , in  $\bar{\alpha}$  and  $\alpha e_1$ .

**Case a1.** For  $\bar{\alpha}e_1\alpha$ , from (11), the corresponding terms of  $e_1e_{h_2}e_{h_3}\cdots e_{h_r}$  with  $1 < h_2 < h_3 < \cdots < h_r$  and  $1 < r \leq n$  in  $\bar{\alpha}e_1 = (\sum_A \alpha_A \bar{e}_A)e_1$  and  $\alpha = \sum_A \alpha_A e_A$  are  $\alpha_{h_2\cdots h_r}(-1)^{\frac{(r-1)(r)}{2}+r-1}e_1e_{h_2}\cdots e_{h_r}$  and  $\alpha_{1h_2\cdots h_r}e_1e_{h_2}\cdots e_{h_r}$ , respectively. Multiplying these terms leads to

$$\begin{aligned} & (-1)^{\frac{(r-1)(r)}{2}+r-1}e_1e_{h_2}\cdots e_{h_r} \cdot e_1e_{h_2}\cdots e_{h_r} \cdot \alpha_{1h_2\cdots h_r} \cdot \alpha_{h_2\cdots h_r} \\ &= (-1)^{\frac{(r-1)(r)}{2}+r-1}(-1)^{\frac{(r)(r+1)}{2}} \cdot \overline{e_1\cdots e_{h_r}} \cdot e_1e_{h_2}\cdots e_{h_r} \cdot \alpha_{1h_2\cdots h_r} \alpha_{h_2\cdots h_r} \\ &= (-1)^{\frac{(r)(r+1)}{2}+r-1+\frac{(r-1)(r)}{2}} \cdot \alpha_{1h_2\cdots h_r} \alpha_{h_2\cdots h_r}. \end{aligned} \quad (13)$$

On the other hand, for  $\bar{\alpha}e_1\alpha$ , from (9), the corresponding terms of  $e_{h_2}e_{h_3}\cdots e_{h_r}$  with  $1 < h_2 < h_3 < \cdots < h_r$  and  $1 < r \leq n$  in  $\bar{\alpha}e_1$  and  $\alpha$  are  $\alpha_{1h_2\cdots h_r}(-1)^{\frac{r(r+1)}{2}+r}e_{h_2}e_{h_3}\cdots e_{h_r}$  and  $\alpha_{h_2\cdots h_r}e_{h_2}\cdots e_{h_r}$ , respectively. Multiplying these terms leads to

$$\begin{aligned} & (-1)^{\frac{r(r+1)}{2}+r}e_{h_2\cdots h_r} \cdot e_{h_2\cdots h_r} \cdot \alpha_{1h_2\cdots h_r} \cdot \alpha_{h_2\cdots h_r} \\ &= (-1)^{\frac{r(r+1)}{2}+r}(-1)^{\frac{(r-1)(r)}{2}} \cdot \overline{e_{h_2\cdots h_r}} \cdot e_{h_2\cdots h_r} \cdot \alpha_{1h_2\cdots h_r} \alpha_{h_2\cdots h_r} \\ &= (-1)^{\frac{r(r+1)}{2}+r+\frac{(r-1)(r)}{2}} \cdot \alpha_{1h_2\cdots h_r} \alpha_{h_2\cdots h_r}. \end{aligned} \quad (14)$$

From (13) and (14), these two terms vanish.

**Case a2.** For  $\bar{\alpha}e_1\alpha$ , from (12), the corresponding terms of  $e_1$  in  $\bar{\alpha}e_1$  and  $\alpha$  are  $\alpha_0e_1$  and  $\alpha_1e_1$ , respectively. Multiplying these terms leads to

$$\alpha_0e_1\alpha_1e_1 = -\alpha_0\alpha_1. \quad (15)$$

On the other hand, for  $\bar{\alpha}e_1\alpha$ , from (10), the corresponding terms of  $e_0$  in  $\bar{\alpha}e_1$  and  $\alpha$  are  $\alpha_1$  and  $\alpha_0$ , respectively. Multiplying these terms leads to  $\alpha_0\alpha_1$ . Combining with (15), these two terms also vanish.

From Cases a1 and a2, one can obtain that the coefficient for  $e_0$  of  $\bar{\alpha}e_1\alpha$  equals zero, i.e.,

$$\langle \tau_{e_0}, \sum_{j=1}^n (\bar{\alpha}e_j\alpha\bar{e}_0) \frac{\partial^2 \varphi}{\partial x_j \partial x_0} \rangle = 0. \quad (16)$$

**Case b1.** For  $\bar{\alpha}\alpha e_1$ , from (11), the corresponding terms of  $e_1e_{h_2}e_{h_3}\cdots e_{h_r}$  with  $1 < h_2 < h_3 < \cdots < h_r$  and  $1 < r \leq n$  in  $\alpha e_1 = (\sum_A \alpha_A e_A)e_1$  and  $\bar{\alpha} = \sum_A \alpha_A \bar{e}_A$  are  $\alpha_{h_2\cdots h_r}(-1)^{r-1}e_1e_{h_2}\cdots e_{h_r}$  and  $\alpha_{1h_2\cdots h_r}\overline{e_1e_{h_2}\cdots e_{h_r}}$ , respectively. Multiplying these terms leads to

$$\begin{aligned} & (\alpha_{1h_2\cdots h_r}\overline{e_1e_{h_2}\cdots e_{h_r}}) \cdot (\alpha_{h_2\cdots h_r}e_{h_2}\cdots e_{h_r} \cdot e_1) \\ &= (\alpha_{1h_2\cdots h_r}\overline{e_1e_{h_2}\cdots e_{h_r}}) \cdot ((-1)^{r-1}e_1e_{h_2}\cdots e_{h_r} \cdot \alpha_{h_2\cdots h_r}) \\ &= (-1)^{r-1}\alpha_{1h_2\cdots h_r} \cdot \alpha_{h_2\cdots h_r}. \end{aligned} \quad (17)$$

On the other hand, for  $\bar{\alpha}\alpha e_1$ , from (9), the corresponding terms of  $e_{h_2}e_{h_3}\cdots e_{h_r}$  with  $1 < h_2 < h_3 < \cdots < h_r$  and  $1 < r \leq n$  in  $\alpha e_1$  and  $\bar{\alpha}$  are  $\alpha_{1h_2\cdots h_r}(-1)^r e_{h_2}\cdots e_{h_r}$  and  $\alpha_{h_2\cdots h_r}\overline{e_{h_2}\cdots e_{h_r}}$ , respectively. Multiplying these terms leads to

$$\begin{aligned} & (\alpha_{h_2\cdots h_r}\overline{e_{h_2}\cdots e_{h_r}}) \cdot (\alpha_{1h_2\cdots h_r}e_1\cdots e_{h_r} \cdot e_1) \\ &= (\alpha_{h_2\cdots h_r}\overline{e_{h_2}\cdots e_{h_r}}) \cdot ((-1)^r e_{h_2}\cdots e_{h_r} \cdot \alpha_{1h_2\cdots h_r}) \\ &= (-1)^r \alpha_{h_2\cdots h_r} \cdot \alpha_{1h_2\cdots h_r}. \end{aligned} \quad (18)$$

From (17) and (18), these two terms vanish.

**Case b2.** For  $\bar{\alpha}e_1$ , from (12), the corresponding terms of  $e_1$  in  $\alpha e_1$  and  $\bar{\alpha}$  are  $\alpha_0 e_1$  and  $\alpha_1 \bar{e}_1$ , respectively. Multiplying these terms leads to

$$\alpha_0 e_1 \alpha_1 \bar{e}_1 = \alpha_0 \alpha_1. \quad (19)$$

On the other hand, for  $\bar{\alpha}e_1$ , from (10), the corresponding terms of  $e_0$  in  $\alpha e_1$  and  $\bar{\alpha}$  are  $-\alpha_1$  and  $\alpha_0$ , respectively. Multiplying these terms leads to  $-\alpha_0 \alpha_1$ . Combining with (19), these two terms also cancel.

From Cases b1 and b2, one can obtain that the coefficient for  $e_0$  of  $\bar{\alpha}e_1\alpha$  equals zero, i.e.,

$$\langle \tau_{e_0}, \sum_{j=1}^n (\bar{\alpha}e_j \bar{e}_0) \frac{\partial^2 \varphi}{\partial x_j \partial x_0} \rangle = 0. \quad (20)$$

Thus,  $I_7 = 0$  from (16) and (20).

To compute  $I_6$ , i.e., to get  $[\bar{\alpha}e_i \alpha \bar{e}_j]_0$  for  $i \neq j$ , similar with the analysis of  $I_7$ , we should divide the vectors in  $\bar{\alpha}e_i$  and  $\alpha \bar{e}_j$  into four cases.

**Case c1.**  $i \in A$ ,  $j \notin A$  for  $e_A$  in  $\bar{\alpha}$  and  $i \notin B$ ,  $j \in B$  for  $e_B$  in  $\alpha$  with  $A-i = B-j$ .

For this case, firstly, we assume  $e_A = e_{h_1 \dots h_{p(i)} \dots h_r}$  and  $h_{p(i)} = i$ ,  $e_B = e_{h_1 \dots h_{p(j)} \dots h_r}$  and  $h_{p(j)} = j$ . We have

$$\begin{aligned} \alpha_A \bar{e}_A e_i &= \alpha_A (-1)^{\frac{r(r+1)}{2}} \cdot e_{h_1} \dots e_i \dots e_{h_r} \cdot e_i \\ &= \alpha_A (-1)^{\frac{r(r+1)}{2} + r - p(i)} e_{h_1} \dots e_i^2 \dots e_{h_r}, \\ &= \alpha_A (-1)^{\frac{r(r+1)}{2} + r - p(i) + 1} e_{A-i}, \\ \alpha_B e_B \bar{e}_j &= \alpha_B e_{h_1} \dots e_j \dots e_{h_r} \cdot \bar{e}_j \\ &= \alpha_B (-1)^{r - p(j)} e_{h_1} \dots e_j \bar{e}_j \dots e_{h_r}, \\ &= \alpha_B (-1)^{r - p(j)} e_{B-j}. \end{aligned}$$

Then

$$\begin{aligned} \alpha_A \bar{e}_A e_i \alpha_B e_B \bar{e}_j &= \alpha_A (-1)^{\frac{r(r+1)}{2} + r - p(i) + 1} e_{A-i} \alpha_B (-1)^{r - p(j)} e_{B-j} \\ &= \alpha_A \alpha_B (-1)^{\frac{r(r+1)}{2} + r - p(i) + 1 + r - p(j) + \frac{r(r-1)}{2}} e_{A-i} e_{B-j} \\ &= \alpha_A \alpha_B (-1)^{r^2 + 1 - p(i) - p(j)}. \end{aligned} \quad (21)$$

**Case c2.**  $i \notin A$ ,  $j \in A$  for  $e_A$  in  $\bar{\alpha}$  and  $i \in B$ ,  $j \notin B$  for  $e_B$  in  $\alpha$  with  $A+i = B+j$ .

We assume  $e_A = e_{h_1 \dots h_{p(j)} \dots h_r}$  and  $h_{p(j)} = j$ ,  $e_B = e_{h_1 \dots h_{p(i)} \dots h_r}$  and  $h_{p(i)} = i$ . We have

$$\begin{aligned} \alpha_A \bar{e}_A e_i &= \alpha_A (-1)^{\frac{r(r+1)}{2}} \cdot e_{h_1} \dots e_j \dots e_{h_r} \cdot e_i, \\ \alpha_B e_B \bar{e}_j &= \alpha_B e_{h_1} \dots e_i \dots e_{h_r} \cdot \bar{e}_j \\ &= -\alpha_B e_{h_1} \dots e_i \dots e_{h_r} \cdot e_j \\ &= \alpha_B e_{h_1} \dots e_j \dots e_{h_r} \cdot e_i. \end{aligned}$$

Then

$$\begin{aligned} \alpha_A \bar{e}_A e_i \alpha_B e_B \bar{e}_j &= \alpha_A (-1)^{\frac{r(r+1)}{2}} \cdot e_{h_1} \dots e_j \dots e_{h_r} \cdot e_i \alpha_B e_{h_1} \dots e_j \dots e_{h_r} \cdot e_i \\ &= \alpha_A \alpha_B (-1)^{\frac{r(r+1)}{2} + \frac{(r+1)(r+2)}{2}} e_{h_1} \dots e_j \dots e_{h_r} \cdot e_i e_{h_1} \dots e_j \dots e_{h_r} \cdot e_i \\ &= \alpha_A \alpha_B (-1)^{r^2 + 1}. \end{aligned}$$

**Case c3.**  $i \in A$ ,  $j \in A$  for  $e_A$  in  $\bar{\alpha}$  and  $i \notin B$ ,  $j \notin B$  for  $e_B$  in  $\alpha$  with  $A-i = B+j$ .

For this case, we assume  $e_A = e_{h_1 \dots h_{p(i)} \dots h_{p(j)} \dots h_{r+2}}$  with  $h_{p(i)} = i$ ,  $h_{p(j)} = j$ . Without loss of generality, we assume  $i < j$ . Furthermore, let  $e_B = e_{h_1 \dots h_r}$ . We have

$$\begin{aligned}
\alpha_A \bar{e}_A e_i &= \alpha_A (-1)^{\frac{(r+2)(r+3)}{2}} \cdot e_{h_1} \dots e_i \dots e_j \dots e_{h_{r+2}} \cdot e_i \\
&= \alpha_A (-1)^{\frac{(r+2)(r+3)}{2} + r+2-h(i)} \cdot e_{h_1} \dots e_j \dots e_{h_{r+2}} \cdot e_i^2 \\
&= \alpha_A (-1)^{\frac{(r+2)(r+3)}{2} + r+1-h(i)} \cdot e_{h_1} \dots e_j \dots e_{h_{r+2}} \\
&= \alpha_A (-1)^{\frac{(r+2)(r+3)}{2} + r+1-h(i) + r+2-h(j)} \cdot e_{h_1} \dots e_{h_{r+2}} \cdot e_j, \\
\alpha_B e_B \bar{e}_j &= \alpha_B e_{h_1} \dots e_{h_r} \cdot \bar{e}_j \\
&= -\alpha_B e_{h_1} \dots e_{h_r} \cdot e_j.
\end{aligned}$$

Then

$$\begin{aligned}
\alpha_A \bar{e}_A e_i \alpha_B e_B \bar{e}_j &= \alpha_A (-1)^{\frac{(r+2)(r+3)}{2} + r+1-h(i) + r+2-h(j)} \cdot e_{h_1} \dots e_{h_{r+2}} \cdot e_j (-1) \alpha_B e_{h_1} \dots e_{h_r} \cdot e_j \\
&= \alpha_A \alpha_B (-1)^{\frac{(r+2)(r+3)}{2} - h(i) - h(j)} \cdot e_{h_1} \dots e_{h_{r+2}} \cdot e_j e_{h_1} \dots e_{h_r} \cdot e_j \\
&= \alpha_A \alpha_B (-1)^{\frac{(r+2)(r+3)}{2} - h(i) - h(j) + \frac{(r+1)(r+2)}{2}} \cdot \overline{e_{h_1} \dots e_{h_{r+2}} \cdot e_j} e_{h_1} \dots e_{h_r} \cdot e_j \\
&= \alpha_A \alpha_B (-1)^{r^2 - h(j) - h(i)}.
\end{aligned}$$

**Case c4.**  $i \notin A$ ,  $j \notin A$  for  $e_A$  in  $\bar{\alpha}$  and  $i \in B$ ,  $j \in B$  for  $e_B$  in  $\alpha$  with  $A+i = B-j$ .

For this case, we assume  $e_A = e_{h_1 \dots h_r}$ ,  $e_B = e_{h_1 \dots h_{p(i)} \dots h_{p(j)} \dots h_{r+2}}$  with  $h_{p(i)} = i$ ,  $h_{p(j)} = j$  and  $i < j$ . We have

$$\begin{aligned}
\alpha_A \bar{e}_A e_i &= \alpha_A (-1)^{\frac{r(r+1)}{2}} \cdot e_{h_1} \dots e_{h_r} \cdot e_i, \\
\alpha_B e_B \bar{e}_j &= \alpha_B e_{h_1} \dots e_i \dots e_j \dots e_{h_{r+2}} \cdot \bar{e}_j \\
&= \alpha_B (-1)^{r+2-h(j)} \cdot e_{h_1} \dots e_i \dots e_{h_{r+2}} \cdot e_j \bar{e}_j \\
&= \alpha_B (-1)^{r+2-h(j) + r+2-h(i)-1} \cdot e_{h_1} \dots e_{h_{r+2}} \cdot e_i \\
&= \alpha_B (-1)^{1-h(j)-h(i)} \cdot e_{h_1} \dots e_{h_{r+2}} \cdot e_i
\end{aligned}$$

Then

$$\begin{aligned}
\alpha_A \bar{e}_A e_i \alpha_B e_B \bar{e}_j &= \alpha_A (-1)^{\frac{r(r+1)}{2}} \cdot e_{h_1} \dots e_{h_r} \cdot e_i \alpha_B (-1)^{1-h(j)-h(i)} \cdot e_{h_1} \dots e_{h_{r+2}} \cdot e_i \\
&= \alpha_A \alpha_B (-1)^{\frac{r(r+1)}{2} + 1-h(j)-h(i)} \cdot e_{h_1} \dots e_{h_r} \cdot e_i \cdot e_{h_1} \dots e_{h_{r+2}} \cdot e_i \\
&= \alpha_A \alpha_B (-1)^{\frac{r(r+1)}{2} + 1-h(j)-h(i) + \frac{(r+1)(r+2)}{2}} \cdot \overline{e_{h_1} \dots e_{h_r} \cdot e_i} \cdot e_{h_1} \dots e_{h_{r+2}} \cdot e_i \\
&= \alpha_A \alpha_B (-1)^{r^2 - h(j) - h(i)}.
\end{aligned}$$



Combining cases c1-c4, we have

$$\begin{aligned}
I_6 &= \langle \tau_{e_0}, \sum_{j \neq i}^n (\bar{\alpha} e_j \alpha \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \rangle \\
&= \langle \tau_{e_0}, \sum_{j \neq i}^n ((\sum_A e_A \bar{\alpha} \alpha_A) e_j (\sum_B e_B \alpha_B) \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \rangle \\
&= \langle \tau_{e_0}, \sum_{j \neq i}^n ((\sum_A e_A \bar{\alpha} \alpha_A) e_i (\sum_B e_B \alpha_B) \bar{e}_j) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \rangle \\
&= \sum_{j \neq i}^n \langle \tau_{e_0}, (\sum_A e_A \bar{\alpha} \alpha_A) e_i (\sum_B e_B \alpha_B) \bar{e}_j \rangle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \\
&= \sum_{j \neq i}^n \langle \tau_{e_0}, (\sum_A e_A \bar{\alpha} \alpha_A) e_i (\sum_B e_B \alpha_B) \bar{e}_j \rangle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \\
&= 2^n \sum_{j \neq i}^n \left( \sum_{i \in A, j \notin A; A-i=B-j} \alpha_A \alpha_B (-1)^{r^2+1-p(i)-p(j)} \right. \\
&\quad + \sum_{i \notin A, j \in A; A+i=B+j} \alpha_A \alpha_B (-1)^{r^2+1} \\
&\quad + \sum_{i \in A, j \in A; A-i=B+j} \alpha_A \alpha_B (-1)^{r^2-h(j)-h(i)} \\
&\quad \left. + \sum_{i \notin A, j \notin A; A+i=B-j} \alpha_A \alpha_B (-1)^{r^2-h(j)-h(i)} \right) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}.
\end{aligned}$$

In all,

$$\begin{aligned}
I_3 &= \int_{\Omega} I_4 e^{-\varphi} dx \\
&= \int_{\Omega} (I_5 + I_6 + I_7) e^{-\varphi} dx \\
&= -2^{n+1} \int_{\Omega} \sum_{i=1}^n \left( \sum_{i \notin A, |A|^2 \text{ is odd}} \alpha_A^2 + \sum_{i \in A, |A|^2 \text{ is even}} \alpha_A^2 \right) \frac{\partial^2 \varphi}{\partial x_i^2} e^{-\varphi} dx \\
&\quad + 2^n \int_{\Omega} \sum_{j \neq i}^n \left( \sum_{i \in A, j \notin A; A-i=B-j} \alpha_A \alpha_B (-1)^{r^2+1-p(i)-p(j)} \right. \\
&\quad + \sum_{i \notin A, j \in A; A+i=B+j} \alpha_A \alpha_B (-1)^{r^2+1} \\
&\quad + \sum_{i \in A, j \in A; A-i=B+j} \alpha_A \alpha_B (-1)^{r^2-h(j)-h(i)} \\
&\quad \left. + \sum_{i \notin A, j \notin A; A+i=B-j} \alpha_A \alpha_B (-1)^{r^2-h(j)-h(i)} \right) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} e^{-\varphi} dx.
\end{aligned}$$

Then

$$\|\bar{D}_{\varphi}^* \alpha\|^2 = \|\bar{D} \alpha\|^2 + \int_{\Omega} |\alpha|_0^2 \Delta \varphi e^{-\varphi} dx + I_3. \quad (22)$$

If  $\frac{\partial^2 \varphi}{\partial x_j \partial x_i} = 0$ ,  $i \neq j$ ,  $1 \leq i, j \leq n$  and  $\frac{\partial^2 \varphi}{\partial x_i^2} \leq 0$ ,  $1 \leq i \leq n$ , we have  $I_3 \geq 0$ , and

$$\|\bar{D}_{\varphi}^* \alpha\|^2 \geq \int_{\Omega} |\alpha|_0^2 \Delta \varphi e^{-\varphi} dx.$$

With the above analysis, we can prove Theorem 12 easily.

*Proof* It is sufficient to prove the theorem if condition (3) in Theorem 11 is presented. By Cauchy-Schwarz inequality in Proposition 26, we have for any  $\alpha \in C_0^\infty(\Omega, \mathcal{A})$  that

$$\begin{aligned} |(f, \alpha)_\varphi|_0^2 &= \left| \int_\Omega \bar{f} \cdot \alpha e^{-\varphi} dx \right|_0^2 \\ &= \left| \int_\Omega \bar{f} \cdot \frac{1}{\sqrt{\Delta\varphi}} \cdot \alpha \cdot \sqrt{\Delta\varphi} \cdot e^{-\varphi} dx \right|_0^2 \\ &\leq \left\| \bar{f} \frac{1}{\sqrt{\Delta\varphi}} \right\|^2 \cdot \left\| \alpha \cdot \sqrt{\Delta\varphi} \right\|^2 \\ &= \int_\Omega \left| \frac{\bar{f}}{\sqrt{\Delta\varphi}} \right|_0^2 e^{-\varphi} dx \cdot \int_\Omega |\alpha \cdot \sqrt{\Delta\varphi}|_0^2 e^{-\varphi} dx \\ &\leq c \|\bar{D}_\varphi^* \alpha\|^2. \end{aligned}$$

The proof is completed with Theorem 11.

It should be noticed that when  $n = 1$ ,  $I_3 = 0$ . Then it comes from equation (22) that the Hörmander's  $L^2$  theorem in  $\mathbb{R}^2$  could be described which equals the classical Hörmander's  $L^2$  theorem in  $\mathbb{C}$ .

**Corollary 41** *Given  $\varphi \in C^2(\Omega, \mathbb{R})$  with  $\Omega$  being an open subset of  $\mathbb{R}^2$ ;  $\Delta\varphi \geq 0$ . Then for all  $f \in L^2(\Omega, \mathcal{A}, \varphi)$  with  $\int_\Omega \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx = c < \infty$ , there exists a  $u \in L^2(\Omega, \mathcal{A}, \varphi)$  such that*

$$\bar{D}u = f$$

with

$$\|u\|^2 \leq \int_\Omega \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx.$$

## 5 Conclusion

In this paper, based on the Hörmander's  $L^2$  theorem in complex analysis, the Hörmander's  $L^2$  theorem for Dirac operator in  $\mathbb{R}^{n+1}$  has been obtained by Clifford algebra. When  $n = 1$ , the result is equivalent to the classical Hörmander's  $L^2$  theorem in complex variable. Moreover, for any  $f$  in  $L^2$  space over a bounded domain with value in Clifford algebra, there is a weak solution of Dirac operator with the solution in the  $L^2$  space as well. The potential applications of the results will be studied in our future work.

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## References

- Brackx et al(1982). Brackx F, Delanghe R, Sommen F (1982) Clifford Analysis, Research Notes in Mathematics. London, Pitman
- De Ridder et al(2012). De Ridder H, De Schepper H, Sommen F (2012) Fueter polynomials in discrete Clifford analysis. Mathematische Zeitschrift 272 (2012) :253–268.
- Gong et al(2009). Gong Y, Leong IT, Qian T (2009) Two integral operators in Clifford analysis. Journal of Mathematical Analysis and Applications 354(2):435–444
- Hörmander(1965). Hörmander L (1965)  $L^2$  estimates and existence theorems for the operator. Acta Mathematica 113(1):89–152
- Huang et al(2006). Huang S, Qiao YY, Wen GC (2006) Real and Complex Clifford Analysis, Advances in Complex Analysis and Its Applications. New York, Springer
- Qian and Ryan(1996). Qian T, Ryan J (1996) Conformal transformations and Hardy spaces arising in Clifford analysis. Journal of Operator Theory 35(2):349–372
- Ryan(1990). Ryan J (1990) Iterated Dirac operators in  $c^n$ . Zeitschrift für Analysis und ihre Anwendungen 9:385–401
- Ryan(1995). Ryan J (1995) Cauchy-Green type formulae in Clifford analysis. Transactions of the American Mathematical Society 347(4):1331–1342
- Ryan(2000). Ryan J (2000) Basic Clifford analysis. Cubo Matemática Educacional 2:226–256